Urysohn Lemma or Luzin-Menshov Theorem?*

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Abstract. This article concerns Urysohn Lemma and the Luzin-Menshov Theorem, logically separated joined however by a mathematical pattern.

Keywords: Urysohn Lemma, Luzin-Menshov Theorem.

2010 Mathematics Subject Classification: 01A60, 26-03.

Introduction

This article concerns the two theorems exposed in the title, logically separated joined however by a mathematical pattern. There is no symmetry in the position of these theorems in mathematics. The importance of Urysohn Lemma lies in applications. It became later famous as a link between the set theoretical methods and the geometric parts of topology. The Luzin-Menshov Theorem is less known for a broad mathematical community. It was never an object of textbooks. It is a key of the subtle study of derivatives, a fascinating subdiscipline of the real analysis. In contrast to the rather mathematically pure Urysohn Lemma, it became on object of broad discussions as a theorem in itself.

The scenery of birth of both theorems is Moscow in the 1920s, the years of rise of the Moscow set theoretical mathematics under Dmitri Egorov and Nicolai Luzin. It was the time of activity of the young disciples of Luzin who called themselves Luzitania, and who were filled with the diversity of characters, mathematical interests and passions, and with the rivalry besides the close cooperation. The charismatic personage of Luzin led to an admiration of the leader, and on the other side, to

* Elaborated and actualized version of article [15] published in 1996 in Polish.

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R. Wituła, D. Słota, W. Hołubowski (eds.), *Monograph on the Occasion of 100th Birthday Anniversary of Zygmunt Zahorski*. Wydawnictwo Politechniki Śląskiej, Gliwice 2015, pp. 191–199.

conflicts. The most known was the one around analytic sets which happened just before the events described in this article.

Bibliographical sources concerning the famous Lemma, although rather rich, have some intriguing gaps. These for the Luzin-Menshov's Theorem consist mostly of gaps, including the authorship, filled later by many unclear apocryphical facts. Luzin-Menshov Theorem preceeds the Urysohn Lemma chronologically as well as in natural evolution of ideas. The theory of real functions was matured earlier than the set topology, and it is in the author's belief that the Luzin-Menshov Theorem played an influential role for the famous Urysohn Lemma. The links between these theorems were never discussed among topologists, as well as among mathematicians working in real analysis. Even in Moscow. The theorems became however worldwide known, and this is the reason for the article presented here.

1. The Urysohn Lemma

A topological space is called normal if for each pair of its subsets $F \subset U$, where F is closed and U is open, a closed subset K of U containing F in its interior can be placed:

$$F \subset \operatorname{int} K \subset K \subset U \tag{1}$$

in symbols.

Not always the normal spaces were described in this way. H. Tietze [17], who systematically discussed the separation conditions, beginning from T_0 through T_2 to the normality, thought the normality as the condition T_4 allowing to separate each two disjoint closed subsets by the disjoint open neighbourhoods. Although the equivalence of Tietze's conditions with condition (1) is a simple logical exercise, the fact of passing to the form (2) was a significant stimulus for further reasonings with normality This reformulation appeared in Urysohn's first posthume 1925 paper [18], the main purpose of which was the theorem asserting that the cardinality of connected normal spaces (having more than one point) is continuum.

Although in the introductory part of paper [18] Tietze's form of normality was used, but in the preparatory part an auxiliary lemma is proved asserting that the condition T_4 implies condition (1). Thus, we see that this fact was for Urysohn significant, even a novelty.

The chain of inclusions (1) was iterated in [18], and Urysohn obtained a well known to the topologists infinite chain of inclusions,

$$F \subset \ldots \subset \operatorname{int} K_r \subset K_r \subset \ldots \subset U, \tag{2}$$

where r runs over the dyadic fractions $k/2^n$ on $[0,1]$. We could expect that the well known continuous function will be written. But it was not the case, as that chain (2) was sufficient for the proof of theorem concerning cardinality of connected normal spaces (having more than one point).

Only in the third annex to this paper, on page 208 in "Trudy" (the collection of Urysohn's works published in two volumes [20]), a function is announced, existence of which gives an answer to the question raised by Maurice Frechet and concerning the possibility to defining on general topological spaces the non-constant continuous function. Then, the following more precise theorem is expressed:

Theorem 1.1. *Let A and B be two closed disjoint subsets of a normal space E. There exists on E a continuous function f such that $f(x)$ is 0 on A, $f(x)$ is 1 on B, and $0 \leqslant f(x) \leqslant 1$ for each x on E;*

called later the **Urysohn Lemma (UL)**. The known to topologists function

$$f(x) = \inf\{r : \ x \in K_r\}, \tag{3}$$

appeared only along the lines of proof of the theorem (concerning cardinality of connected normal space).

A surprise for the reader, who knows the further development of topology, is that the result remained without applications, treated as an end in itself. Although the problem raised by Frechet was of great general importance for the set topology as a mathematical discipline, it was however without any impact onto the concrete mathematical question.

In the "Additional remarks" at the end of paper [18] Urysohn wrote that "[...] the theorem of paragraph 25 is significant for the problem of metrization, and my aim is in the nearest future to publish a paper in which I shall show that each normal space with a countable base is homeomorphic to a metric space". Let us note an emotional P.S. Alexandrov's comment published many years later in "Trudy" ([20], in comment 6 on page 216) where we read: "In fact, in these lemmas [...] a key to the proof of the metrization theorem is contained".

The paper [18] was finished in August 1924 (as we can read from Alexandrov's comments in "Trudy" ([20], pages 177–218)) three days before Urysohn's tragic death.

Only in the second posthume Urysohn's paper [19], where the Lemma was explicitly stated and proved again, the theorem on the metrizability of normal spaces with countable bases was stated and proved in a well known manner with the use of the Urysohn function.

This second posthume Urysohn's paper, as we can read from Alexandrov's comments to the paper in "Trudy" [20], was elaborated almost entirely (excluding the introductory paragraph) by Alexandrov from dispersed Urysohn's notes. Knowing Alexandrov's further results in set topology, we can include P.S. Alexandrov to the fathers of the Lemma.

2. Luzin-Menshov Theorem

In the theory of real functions we can hear and read, without quotation to any concrete paper, about Luzin-Menshov Theorem. The theorem concerns the sets of density points of measurable sets on the real line. In its essential form it claims that

Theorem 2.1 (LMT). *If F is a perfect subset of a measurable subset U consisting exclusively of its points of density, then there exists a perfect set K lying between F and U such that the set F is contained in the set K^* consisting of density points of K:*

$$F \subset K^* \subset K \subset U, \tag{4}$$

in symbols.

Recall, that p is a density point of a measurable set A if the quota of the measure of set A in the interval $[p-h, p+h]$ with respect to the longitude $2h$ of this interval tends to 1 if h tends to zero. According to Lebesgue, almost all points of any measurable set are its density points. The operation of passing from a measurable set A to its "measure interior" A^* has the same formal properties as the operation of the interior in topological sense. The sets U consisting exclusively of their density points stand in analogy to topologically open sets: we call them as the measure open sets.

Topologically open sets are measure open. However, there are measure open sets not open topologically; for instance, the set of irrationals. Henceforth, not every measure closed set, for instance, the set of rationals, is closed topologically. One point set can be placed in the role of F in (4). Thus, LMT does not assert the normality of the measure topology.

Only about fifty years later this measure topology appeared in real analysis again; see C. Goffman and D. Waterman [7]. It is called recently the density topology of reals. We do not claim that the density topology was the Luzin's and Menshov's aim. The notion of topological space as the set and topology seems to be far from the interests of both mathematicians.

The theorem was stated, but – as we can suppose – remained without proof. The proof was presented by Vera Bogomolova in her 1924 paper [4]. Bogomolova wrote that "the theorem was proved by N.N. Luzin and D.E. Menshov. I did not know their method, I got later another proof, which I am presenting in this paper". The proof of Luzin-Menshov Theorem, given by Bogomolova is far from obviousness. The author is indebted to Iwona Krzemińska [9] who read that paper and delivered to the author the comments of essential value.

Motivations of Bogomolova's paper went beyond the statement (LMT). The theorem was stated and proved in the paper in a special form allowing to get a general approach to particular constructions of the known at that time singular everywhere differentiable functions, for instance that one with densely situated set of intervals of constancy constructed by S. Mazurkiewicz [14]. The problem, as we can read in [4], was suggested by Luzin via his recent interests in A. Denjoy's constructions and his talks with W. Sierpiński in Moscow in the years 1915–1918.

Let us return now to the main thread, in the further part of the paper [4] an appropriate corollary (LMT) was derived. Next the chain (4) of inclusions was iterated and Bogomolova obtained an infinite chain of inclusions

$$F \subset \ldots \subset K_r^* \subset K_r \subset U, \tag{5}$$

where r runs over dyadic fractions on $[0, 1]$. Then, the function

$$f(x) = \inf\{r : \ x \in K_r^*\} \tag{6}$$

was defined with values between 0 and 1, which is 0 on F and 1 outside U. It occurred to be approximately continuous, it means that each point lies as a density point of a measurable set, on which the function is continuous at this point when restricted to this set – see I.P. Natanson [16] for references – thus, the function occurs to be continuous in the sense of measure topology. Being bounded, it is the derivative of its Lebesgue indefinite integral. This everywhere differentiable integral, depending on the manner specializations, occurs to be the Mazurkiewicz function, or nowhere monotone everywhere differentiable function, the last distributed by Bogomolova to A. Denjoy.

3. Comments

The facts. The patterns of functions written by Bogomolova and Urysohn are the same. The Luzin-Menshov Theorem (LMT) serves as a pattern for the notion of normality in Urysohn's research. Bogomolova published the result a year earlier.

On the LMT

It seems that LMT became known to mathematicians by oral communications and was an existing result before Bogomolova's paper It was treated as a final result. Bogomolova's function was the first its known application. For many years the LMT remained without continuation.

The earliest quotations to this theorem, known to the present author, are the papers by Isaiah Maximoff and Zygmunt Zahorski published about twenty years later.

Chronologically first were the papers of Maximoff (1940) devoted to the approximately continuous functions. According to the famous result of the papers published in Kazan [12] and Tôhoku [13], the domain of each Darboux First Class function can be reparametrized by a self-homeomorphism such that the function became approximately continuous; thus it is a derivative if the function is bounded. The tool was the Luzin-Menshov Theorem, which was proved by Maximoff again. In one of this series of Maximoff's paper "V. Bogomoloff" is quoted in a footnote.

The paper [22] by Zahorski published in Tôhoku (1941) begins with quotation of the Luzin-Menshov Theorem without indicating any particular paper. Zahorski presented a proof of the theorem at about five pages in print. It seems that the theorem was known to Zahorski only from hearing and no previous proof was available for him. The theorem was applied to the construction of everywhere differentiable function with given in advance a G-delta set of measure zero where the derivative is infinite, solving in this way a problem raised by Vojtěch Jarnik from Tôhoku 1933. The paper of Bogomolova was not quoted. However, in the same year in Zahorski's paper [23] published in Matematiceskii Sbornik, where the Luzin-Menshov Theorem was discussed again, there is an adnotation in a footnote to Bogomolova as the author of the proof. These two papers of Zahorski were sent from Lvov before June 1941 and reached Tôhoku and Sbornik in July this year. His interest to the singular derivatives is dated at his Warsaw years before the World War II.

A direct relationship with the content of Bogomolova's paper – however isolated in the literature – has the paper by Kaplan and Slobodnik [8] (1976), where the Luzin-Menshov Theorem was stated and proved, together with Bogomolova's function (6) as a corollary. The results of Zahorski's papers are discussed and new consequences from LMT, concerning the non-constant everywhere differentiable functions whose derivatives vanish at the dense set of points, are derived.

Mazurkiewicz function was one of many other functions of this kind of singularity. The first one was given by A. Koepcke (1889), who constructed an everywhere differentiable nowhere monotone function (attributed by Bogomolova to Denjoy). In 1907 D. Pompeiu constructed a strictly increasing function, whose everyhere existing derivative vanishes at the dense set of points. Motivation for these constructions came from the theory of integral. The derivatives of these functions are not Riemann integrable and they signalized the insufficiency of the Riemann integral for restoring functions from derivatives. The classical constructions were made according to the individual methods.

Zahorski's paper [22] initiated a method of reparametrization of the domains of functions of bounded variation by means of a homeomorphisms constructed by him via Luzin-Menshov Theorem, making these functions everywhere differentiable. More explicitly this was done by Zahorski in his 1950 paper [24], where much more general approach to Bogomolova's function (however without any quotations to [4]) is developed. For instance, the function of Mazurkiewicz can be obtained by a reparametrization of the domain of the well known Cantor- Lebesgue staircase function. This Zahorski's approach was developed into a general procedure in 1970's in the book by A. Bruckner [5]. In this book a contemporary proof of LMT is given, however without comments to the the source of the theorem.

In spite of great mathematical value of the results based on LMT, as well as the value of the theorem itself, the results based on it are treated as a kind of mathematical art. The theory of real functions, in particular the subtle results enlighting the nature of the first derivative, never pretended to be dominating ones in mathematics. On the other hand, the questions concerning the derivatives led to a subtle analysis of the mathematical nature of LMT. A subsequent discussion of the theorem led to some interesting generalizations.

On the UL

The fate of the sisterly result, called the Urysohn Lemma, was quite different. We cannot imagine the set topology, thought by Tietze in his 1923 paper, without non-constant continuous real valued functions, that is without a link to spaces of geometrical nature. The pattern, to which the Urysohn function underlies, is of no less importance. Due to it there are diversity of situations, to which the Lemma can be applied joining the set topology with geometry. In the second posthume Urysohn paper [19], prepared for publication by Alexandrov, the Lemma was applied in the proof of the theorem concerning the metrization of normal spaces with countable bases. It is not the aim of this article to describe the other famous theorems of metrizability of normal spaces satisfying much more weaker assumptions, in the proofs of which UL plays a key role. Let us recall only the method of maps into the nerves of open covers, a tool for approximation of spaces by polyhedra, being a key to the Lebesgue dimension theory.

According to a comment of P.S. Alexandrov in “Trudy” ([20], pages 214–218), Urysohn presented his results at the Moscow Mathematical Society in May 1924. The paper by Bogomolova was probably just printed at that time, since it was accepted in Matematiceskii Sbornik in May 1923. Thus, it is difficult to explain the absence of quotations to Bogomolova in Urysohn’s papers. Can we accept as a justification that in June 1923 both P.S.-es went to Goettingen – see the book [3] by M. Becvarova and I. Netuka – being probably far from the Moscow events?

Looking to Bogomolova’s paper, we see that its essence lies in the proof of (4). Namely, this result is called by subsequent authors as the Luzin-Menshov Theorem. The function (6) was treated rather as a forehand expectation. So, UL borrowed from LMT not so much, namely, the stimulus for understanding the normality in form (1). However, we know how important such subtle stimulations can be. What is more, a handsome gesture from the side of Urysohn would be on place.

Looking for the subsequent papers from 30’s, we observe a remarkable silence around the Urysohn Lemma in Alexandrov’s publications. There is a lack of quotation to Urysohn in “Topologie I” written in 1935 jointly with Heinz Hopf, although the Lemma is formulated and applied. In Alexandrov’s and Urysohn’s mathematical CV’s there are no stories concerning the events around the discovery of this so important result in the set topology, in contrast to the stories concerning Urysohn’s results in the theory of dimension.

The situation looks quite different in the Alexandrov’s textbooks [1] (1948) and [2] (1970), written many years later. The Urysohn reformulation (1) of normality is called there as the “Small Urysohn Lemma”, and the formula for the function as the “Great Urysohn Lemma”. Can we explain this change as the result of evaluation of UL caused by the highly valuated at these times the expansive discipline called the general topology?

4. Beyond mathematics

The author has no right to discuss non-mathematical causes of the absence of quotations to LMT in Urysohn’s papers. Perhaps Urysohn and Alexandrov simply overlooked Bogomolova’s results and even LMT itself. This can be confirmed by the fact that after years Alexandrov could not recognize LMT in the “Small Urysohn Lemma”. Perhaps he was never interested in the measure theoretic problems.

However, the absence of quotations can be caused by the coldness of relations of Urysohn and Alexandrov with their mother center in Moscow. In the case of Alexandrov we can say simply about a conflict with Luzin initiated about 1918 around the analytic sets discovered by Souslin.

The years of this story coincide with the decline of the famous Luzitania, a group of young mathematicians gathered around Luzin since 1920 when “the Civil Wars” end. From year to year the young mathematicians, to whom both “PS-es” belonged, wanted to be independent in choosing the problems. This was a great defeat of Luzin caused by his difficult character. Let us quote Alexandrov’s words uttered years later: “The key of Luzin’s tragic fate was his personality, concentrated on his own, distant from people, in his not easy, also for his disciples, complicated psychology". For the story of Luzitania the reader should be addressed to the book by Smilka Zdravkovska [25].

In the late 1920s the mathematical events in Moscow were included into the stream of affairs having sources beyond the pure mathematical life. They began with the general plan of restoration of the Academy of Sciences. Luzin was removed to the philosophical branch of the Academy. Although there were political reasons of that decision, yet, as we know, even Dmitri Egorov, the supervisor and close Luzin's friend, was far from supporting Luzin.

The "travla" – a Russian difficult to translate word – around Luzin came to the apogeum in middle 30's, when there appeared in press many anonymous accusations, among them about the weak doctoral dissertations having been made under Luzin's supervision. Although Luzin did not share the fate of Egorov, who was exiled to Kazan and died there, the events moved Luzin outside of the active mathematical life. These accusations did not have their source in the mathematical community. It seems that the mathematical community was involved in that affair without their own will. The events were described with care by A.P. Yushkevich [21] in a broad political context. Recently there were published the materials from the session of the Academy commission discussing the so called "Luzin affair" [6].

Looking on the published list of Luzin's disciples, only Bogomolova stopped the mathematical search by publishing only one paper, namely her doctoral dissertation. This dissertation was of a great mathematical value, so there is no reason to suggest that it was one of "these weak doctoral theses" written under Luzin's supervision. However, Bogomolova is placed on the list of Luzin's students (available in google) without any personal data, like a non existing person. In the volume "Forty years of Soviet mathematics" [17] the paper of Bogomolova is quoted, but her name on the list of persons at the end of the volume is absent. She is not mentioned among Luzin's disciples, neither in the Luzin's biography in "The Integral and the Trigonometrical Series" written by Nina Bari in 1951. However, she was an existing person, mentioned in the article written by L.A. Lusternik [10] on the occasion of the trip of Moscovian mathematicians to Petrograd in 1921. But, a more intriguing fact is that in the mentioned Luzin's biography, as well as in other biographical articles on Luzin, the Luzin-Menshov Theorem was never noticed, even as Luzin's impact into mathematics. On the other hand, Luzin's 1912 Paris paper devoted to approximately continuous functions is quoted in all textbooks.

It is natural to ask, why the mentioned here facts were so long indifferent to the mathematical community? The UL was so famous and the LMT was till last decades highly elaborated by the real analysists. The indifference to UL can be explained by the fact that the Urysohn Lemma became famous only after the finalization of notions of the general topology with apogeum of applications in the 1960's. The silence around LMT among the real analysists is not so easy. It can be explained by the fact that mathematicians of the generations of years 1920–1940 knew the truth and regarded it as their own. But how to explain the silence among mathematicians of further generations? A bit easier is to understand the silence in Moscow. For Moscow mathematicians, unwillingly involved around 1930 into political conflict, there was no interest to discuss in the dangerous situation one more (overlooked by officials) mathematical enigma which, as we believe, was rather of a pure mathematical character.

Bibliography

1. Aleksandrov P.S.: *Vvedeniie v obshchuiu teoriu mnoshestv i funkcii.* Gosudarstv. Izdat. Techn.-Teor. Lit, Moscow 1948.
2. Aleksandrov P.S.: *Vvedeniie v teoriu mnoshestv i obshchuiu topologiu.* Nauka, Moscow 1977.
3. Becvarova M., Netuka I.: *Jarnik's notes of the lecture course Punktmengen und reelle Funktionen by P. S. Aleksandrov (Göttingen 1928).* Matfyzpress, Prague 2010.
4. Bogomolova W.S.: *Ob odnom klassie funkcij wsudu assimptoticeski neprerywnych.* Mat. Sb. **32** (1924), 152–171.
5. Bruckner A.: *Differentiation of Real Functions,* second edition. CRM Monograph Series **5**, AMS, Providence 1994.
6. *Delo akademika Nikolaia Nikołaevica Luzina – under the editorship of S.S. Demidov and B.V. Levshin* (a manuscript). Institut Istorii Estestvoznaniya i Tekhniki, Arkhiv Rossijskoj Akademii Nauk, Sankt Petersburg 1999.
7. Goffman C., Waterman D.: *On approximately continuous transformations.* Proc. Amer. Math. Soc. **12** (1961), 116–121.
8. Kaplan L.I., Slobodnik S.G.: *Ob odnom klassie funkcij wsiudu assimptoticzeski neprerywnych.* Mat. Zametki **22** (1977), 859–871.
9. Krzemińska I.: *Osobliwości funkcji różniczkowalnych.* Zesz. Nauk. WSI w Opolu, Mat. **13** (1994), 27–42.
10. Lusternik L.A.: *Molodost' moskovskoi matematiceskoi shkoly. Belyie noci (The youth of Moscow Mathematical School. White Nights).* A manusccript.
11. *Matematika v SSSR za sorok let (Forty years of Soviet mathematics).* Moscow 1957.
12. Maximoff I.: *Sur la transformation conntinue de fonctions.* Bull. Soc. Phys. Math. Kazan **12**, no. 3 (1940), 57–81 (in Russian).
13. Maximoff I.: *On density points and approximately continuous functions.* Tôhoku Math. J. **47** (1940), 237–250.
14. Mazurkiewicz S.: *Konstrukcja funkcji różniczkowalnej mającej wszędzie gęsty zbiór przedziałów stałości.* Prace Mat.-Fiz. **27** (1916), 193–201.
15. Mioduszewski J.: *Lemat Urysohna czy twierdzenie Łuzina-Mieńszowa?* Zesz. Nauk. PŚl., Mat.-Fiz. **76** (1996), 141–150.
16. Natanson I.P.: *Teoria funkcij veshchestvennoj peremennoi.* Gosudarstv. Izdat. Techn.-Teor. Lit., Moscow 1950 (*Theory of Functions of Real Variable*, Frederick Ungar Publishing Co., New York 1955).
17. Tietze H.: *Beitrage zur allgemeine Topologie I. Axiome fur verschiedene Fassungen der Umgebungsbegriff.* Math. Ann. **88** (1923), 290–312.
18. Urysohn P.: *Uber die Machtigkeit der zusammenhangenden Menge.* Math. Ann. **94** (1925), 262–295 (also in [20], pp. 177–218).
19. Urysohn P.: *Zur metrizations problem.* Math. Ann. **94** (1925), 309–315 (also in [20], pp. 740–746).
20. Uryson P.S.: *Trudy po topologii i drugim oblastyam matematiki*, t. I–II. Gosudarstv. Izdat. Techn.-Teoret. Lit., Moscow 1951.
21. Yushkevich A.P.: *The N.N. Luzin affair.* Vestnik Akad. Nauk SSSR **4** (1989), 102–113.
22. Zahorski Z.: *Über die Menge der Punkte in welchen die ableitung unendlich ist.* Tôhoku Math. J. **48** (1941), 321–330.
23. Zahorski Z.: *O mnozhestwie tocek niedifferenciruemost'i funkcij (On the set of points of non-differentiability of a continuous function).* Mat. Sbornik, N. Ser. **9(51)** (1941), 487–510.
24. Zahorski Z.: *Sur la première dérivée.* Trans. Amer. Math. Soc. **69** (1950), 1–54.
25. Zdravkovska S., Duren P.L.: *Golden Years of Moscow Mathematics.* AMS, Providence 1993.